Exchangeable Equilibria

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Introduction

Outline

- Games
- Nash and correlated equilibria
- Symmetric games and equilibria
- Definition and interpretation of exchangeable equilibria
- Complete positivity and double nonnegativity
- Another interpretation of exchangeable equilibria
- Exchangeability and de Finetti's theorem
- Properties and examples of exchangeable equilibria
- Computation
- Extensions

What's new

- Everything about exchangeable equilibria

Games

Data

- $n < \infty$ players
- Strategy (or action) set C_i for each player i
- $|C_i| = m$ strategies per player
- Outcomes (or strategy profiles): $C_1 \times \cdots \times C_n$
- Utility (or payoff) function $u_i : C_1 \times \cdots \times C_n \to \mathbb{R}$ for each i

Interpretation

- The data is common knowledge: each player knows it, knows his opponents know it, etc.
- Simultaneously, each player i chooses action $s_i \in C_i$
- Each player wants to maximize his expected utility given his knowledge of others' actions

Example

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

Fitting into the framework

- $n = 2$ players
- Player 1 chooses rows
- Player 2 chooses columns
- $m = 2$ strategies per player
- $C_1 = C_2 = \{\text{Wimpy}, \text{Macho}\}$
- Cell (s_1, s_2) contains utility pair $(u_1(s_1, s_2), u_2(s_1, s_2))$

Solving games

Solution concepts

- (How) can we *describe* or *prescribe* how to play?
- Many existing notions of "reasonable" behavior in games
- Each makes assumptions about players
- Stronger assumptions ⇒ stronger predictions
- Sad truth: no single "best" / "right" solution concept

Equilibria

- Of these, only Nash (NE) and correlated equilibria (CE) today
- CE: Outcome distributions stable under unilateral deviations
- NE: CE in which players choose strategies independently
- XE: ? (wait a few slides)

Chicken – Nash equilibria

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

Nash equilibria

- All three equilibria in three notations

Tuple	(M, W)	(W, M)	$(\frac{1}{2}W + \frac{1}{2}M, \frac{1}{2}W + \frac{1}{2}M)$
Product	$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
Joint law	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Chicken – correlated equilibria

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4, 4)$	$(1, 5)$
Macho	$(5, 1)$	$(0, 0)$

Correlated equilibria

- Example correlated equilibria (joint laws)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Chicken – correlated equilibrium conditions

The game of chicken

(u_1, u_2)	Wimpy	Macho
Wimpy	$(4,4)$	$(1,5)$
Macho	$(5,1)$	$(0,0)$

$$(X, Y) \sim D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Incentive constraints

- For example if the row player receives recommendation $X = M$ he cannot expect to improve by playing W instead:

$$\mathbb{E}(u_1(M, Y) \mid X = M) \geq \mathbb{E}(u_1(W, Y) \mid X = M)$$

$$5\frac{c}{c+d} + 0\frac{d}{c+d} \geq 4\frac{c}{c+d} + 1\frac{d}{c+d} \qquad (c + d > 0)$$

$$5c + 0d \geq 4c + 1d$$

- Linear inequalities: $b, c \geq a, d$

Properties of equilibria

Correlated equilibria (CE) [Aumann]

- Polytope: m^n nonnegative vars, $\mathcal{O}(nm^2)$ linear inequalities
- Rational utilities $\Rightarrow$ rational extreme points (vertices)
- Existence via minimax / duality / separating hyperplanes [HS]
- Easy to compute (solve a linear program)
  - Even without fixing n [Papadimitriou]

Nash equilibria (NE)

- Correlated equilibria which are independent distributions
- Generically finitely many (odd number) [Wilson]
- Two players: rational, lie at extreme points of CE [ER,C]
- More players: may be irrational [Nash]
- Existence via fixed point theorems [Nash]
- Hard to compute ("PPAD-complete") [DGP,CDT]

Symmetric games

Definition

- For this talk a **symmetric** game will be a game satisfying
  - Common strategy space $C_1 = \ldots = C_n$
  - Permuting actions permutes utilities in the same way
- (Many results hold with a more general definition)

The idea

- Labels of players don't matter
- n booths – each has m buttons labeled by C_1, output slot
  - Each player assigned booth
  - Selects action, receives payoff from the slot
- It doesn't matter who uses which booth
- Two-player case: utility matrices satisfy $B = A^T$
  - e.g. chicken
- From now on, all games symmetric

Symmetric equilibria

Symmetric correlated equilibria

- If $(X_1, \ldots, X_n)$ are distributed according to a CE, so are $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for any permutation σ
  - Two-player case: $W \in \text{CE} \Rightarrow W^T \in \text{CE}$
- Symmetric correlated equilibria (CE_{Sym}): fixed by all σ
  - Two-player case: $W = W^T$
- Existence: take any CE, average over all permutations

Symmetric Nash equilibria

- Independent symmetric correlated equilibria
- i.i.d. correlated equilibria

Properties

- Basically same as asymmetric equilibria

Symmetric players

Interpretation of symmetric games

- Restricting to symmetric games is a strong condition
- All players have the same preferences
- It is hard to imagine this happening "by accident"
- If assuming this, we might as well add:

Clone assumption

- Players are "clones": identical decision-making agents
- Same information $\Rightarrow$ same decision

Discussion

- Natural symmetry assumption– why go halfway?
- Weaker-sounding Bayesian equivalent later

Hidden variable interpretation

Implication of clone assumption

- Suppose there is no explicit correlating device
- Players base actions on knowledge of state of the world
- Clone assumption: players make independent measurements of the state, interpret these in the same way
- Conclusion: actions i.i.d. conditioned on state of the world
- Otherwise symmetry would implicitly be broken

Definition

- A correlated equilibrium of a symmetric game which is i.i.d. conditioned on some hidden parameter is called an **exchangeable equilibrium** (XE).

Complete positivity

Definition

- The set of $m \times m$ **completely positive** matrices is

$$\mathrm{CP}_m^n = \mathrm{cone}(\mathrm{i.\,i.\,d.}).$$

- For two players: $\mathrm{CP}_m^2 = \mathrm{conv}\{xx^T \mid x \in \mathbb{R}_{\geq 0}^m\}$

Properties

- Random variables are i.i.d. conditioned on a parameter if and only if their joint distribution is completely positive
- So $\mathrm{XE} = \mathrm{CE} \cap \mathrm{CP}_m^n$
- CP_m^n is a closed convex cone, i.i.d. distributions extreme
- $X \in \mathrm{CP}_m^2$ and $X_{ij} > 0$ implies $X_{ii}, X_{jj} > 0$
  - Proof: one of the terms xx^T has $x_i, x_j > 0$
- So e.g. $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \notin \mathrm{CP}_2^2$

Double nonnegativity

Observation

- $x \in \mathbb{R}^m_{\geq 0} \Rightarrow xx^T$ symmetric, elementwise nonnegative
- $y \in \mathbb{R}^m \Rightarrow y^T x \in \mathbb{R}$ and $y^T xx^T y = (y^T x)^2 \geq 0$
- So xx^T is positive semidefinite

Definition

- A matrix is **doubly nonnegative** (DNN^2_m) if it is symmetric, elementwise nonnegative, and positive semidefinite
- (More complicated definition for DNN^n_m)

Properties

- DNN^n_m is convex so $\mathrm{CP}^n_m \subseteq \mathrm{DNN}^n_m$
- Equality if and only if $n = 2$ and $m \leq 4$ or $m = 2$
- Semidefinite representable

Chicken – exchangeable equilibria

Computation

- $\mathsf{XE} = \mathsf{CE} \cap \mathsf{CP}_2^2 = \mathsf{CE} \cap \mathsf{DNN}_2^2$
- An exchangeable equilibrium looks like $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with

$$\begin{aligned} a, b, c, d &\geq 0 && \text{(nonnegativity)} \\ a + b + c + d &= 1 && \text{(normalization)} \\ b, c &\geq a, d && \text{(incentives)} \\ b &= c && \text{(symmetry)} \\ ad &\geq bc && \text{(semidefiniteness)} \end{aligned}$$

- If any incentive constraint were not tight then $b = c > 0$ and $bc > ad$, contradicting semidefiniteness
- So $a = b = c = d = \frac{1}{4}$
- Unique exchangeable equilibrium: the symmetric Nash equil.

Unknown opponent interpretation

Thought experiment

- Pick $N \gg n$ people, ask each how he would play the game
- Result: a sequence $(X_1, \ldots, X_N)$ of elements of C_1

Bayesian observer's prior for $(X_1, \ldots, X_N)$

- Bayesian ignorance: distribution of $(X_1, \ldots, X_N)$ is the same as that of $(X_{\sigma(1)}, \ldots, X_{\sigma(N)})$ for any permutation σ
- Observer believes players are rational
- Distribution of $(X_1, \ldots, X_n)$ must be in CE [Aumann]

Consequences

- Distribution of $(X_1, \ldots, X_n)$ is in CE_{Sym}
- Can we say more?

Definition

- The distribution of a random sequence $(X_1, X_2, \ldots)$ is **exchangeable** if invariant under permuting finitely many X_i.

Properties

- i.i.d. sequences obviously exchangeable
- Convexity: conditionally i.i.d. $\Rightarrow$ exchangeable

De Finetti's Theorem

- Exchangeable $\Rightarrow$ conditionally i.i.d. on some parameter

Conclusion

- Acceptable priors as $N \to \infty$ are the exchangeable equilibria

Exchangeable equilibria

Properties

- XE is compact, convex, semialgebraic, not generally polyhedral
- Existence: add symmetry to [HS] minimax argument
- Sandwiched between symmetric Nash and correlated equilibria

$$\operatorname{conv}(\mathrm{NE}_{\mathrm{Sym}}) = \operatorname{conv}(\mathrm{CE} \cap \mathrm{i.\,i.\,d.}) \subseteq \mathrm{CE} \cap \mathrm{CP}^n_m = \mathrm{XE} \subseteq \mathrm{CE}_{\mathrm{Sym}}$$

- $\operatorname{conv}(\mathrm{NE}_{\mathrm{Sym}}) = \mathrm{XE}$ if $m = n = 2$, can be strict otherwise
- $\mathrm{NE}_{\mathrm{Sym}}$ contained in extreme points of XE
- $\mathrm{NE}_{\mathrm{Sym}} \subsetneq \mathrm{NE} \Rightarrow \mathrm{XE} \subsetneq \mathrm{CE}_{\mathrm{Sym}}$

Separation example

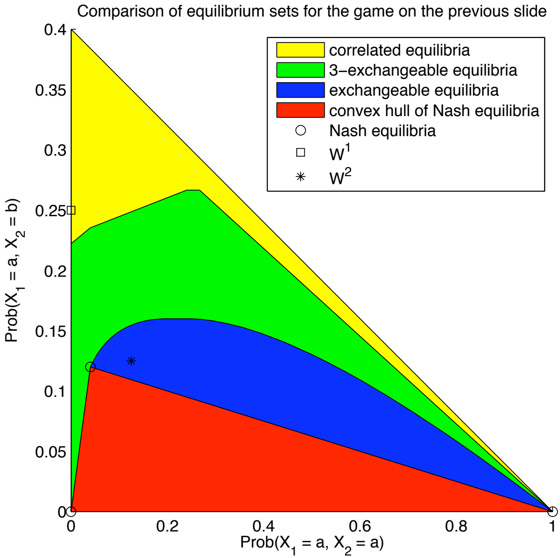
Example game

(u_1, u_2)	a	b	c
a	$(5,5)$	$(5,4)$	$(0,0)$
b	$(4,5)$	$(4,4)$	$(4,5)$
c	$(0,0)$	$(5,4)$	$(5,5)$

- Symmetric Nash equilibria:
  $\begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1/5 & 3/5 & 1/5\end{bmatrix}$
- Non-exchangeable correlated equilibrium: $W^1 = \begin{bmatrix}0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0\end{bmatrix}$
- Exchangeable equilibrium not in $\mathrm{conv}(\mathrm{NE}_{\mathrm{Sym}})$:

$$W^2 = \begin{bmatrix}\frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\frac{1}{2} \\ \frac{1}{2} \\ 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} \\ \frac{1}{2} \\ 0\end{bmatrix}^T + \frac{1}{2}\begin{bmatrix}0 \\ \frac{1}{2} \\ \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 \\ \frac{1}{2} \\ \frac{1}{2}\end{bmatrix}^T$$

Separation example, plotted



- Correlated equil. which is not exchangeable:

$$W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

- Exchangeable equil. not in conv(Nash):

$$W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Three player example

Don't be greedy

- $C_1 = C_2 = C_3 = \{0, 1\}$

$$u_i(s_1, s_2, s_3) = \begin{cases} 0 & \text{when } s_1 = s_2 = s_3 = 1 \\ s_1 + s_2 + s_3 & \text{otherwise} \end{cases}$$

- Symmetric Nash equilibria are Bernoulli(p) for some p
- Algebra: only solution is $p^* = \frac{1}{\sqrt{3}}$
- $\mathrm{XE} = \mathrm{CE} \cap \mathrm{CP}_2^3 = \mathrm{CE} \cap \mathrm{DNN}_2^3$
- More algebra: $\mathrm{XE} = \mathrm{NE}_{\mathrm{Sym}}$
- Unique exchangeable equilibrium, irrational probabilities

Obstacles

- Can we compute exchangeable equilibria efficiently?
- With rational arithmetic, we must accept some error
- Can we approximate exchangeable equilibria efficiently (say in polynomial time in the input size and desired precision)?
- Can replace CP_m^n with DNN_m^n to get SDP relaxation
  - Exact if $m = 2$ or $n = 2$ and $m \leq 4$
  - Otherwise, no performance guarantee
- Checking if there exists a completely positive matrix approximately satisfying *one* given linear inequality is NP-hard
- Perhaps the correlated equilibrium constraints are easy?

Computing exchangeable equilibria #2

Solution

- [PR] cleverly apply ellipsoid method to implement [HS] existence proof; intended for large games
- Idea: symmetrize algorithm in same way as proof?
- Paradox: output should be exact XE, but is rational
- Resolution: gap in arithmetic precision analysis in [PR]
- Fix gap: approximate exchangeable equilibrium algorithm polynomial in input and # bits of precision
- Later, [JLB] show how to break symmetry, get exact CE

[PR] algorithm sketch

Dual problems

(P)	$\max \sum x_s$	$\min 0$ (D)
	m^n vars $x_s \geq 0$ $(s \in \prod_i C_i)$	m^n constraints
	$\mathcal{O}(nm^2)$ incentive constraints	$\mathcal{O}(nm^2)$ vars $y_i^{s_i,t_i} \geq 0$

The idea

- Existence of CE $\Leftrightarrow$ (P) unbounded $\Leftrightarrow$ (D) infeasible
- Use ellipsoid method on (D) to show infeasibility
- For any y, need a cut: mixture of constraints violated at y
- [HS] oracle gives cut in product form $x_s = x_{s_1} \cdots x_{s_n}$
- After enough cuts we know dual is infeasible
- Some mixture of these cuts is nonzero, primal feasible

The problem and fix

Changes to compute XE

- Symmetric game $\Rightarrow$ i.i.d. cut
- Any mixture of cuts is completely positive

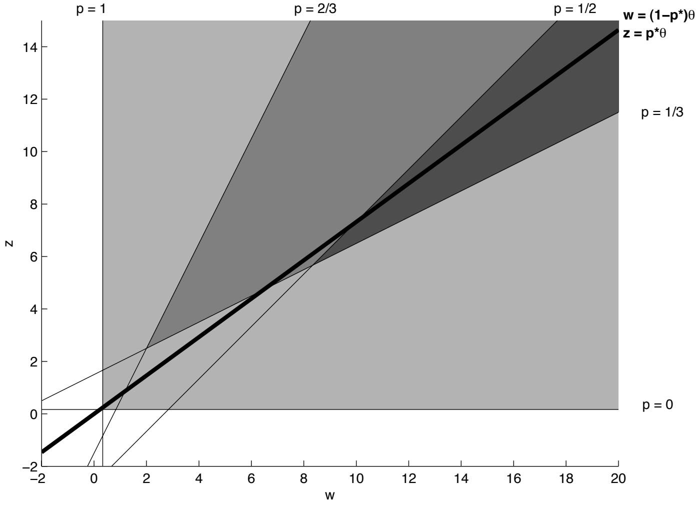
The problem

- Any finite # of (rational) such cuts is jointly feasible
- Finitely many iterations only show solutions of (D) are large

The solution

- This means some mixture of cuts is almost feasible for (P)
- Can compute approximate exchangeable equilibria efficiently

Illustration of feasibility



Extensions of exchangeable equilibria

Observation

- Exchangeable equilibria have a simple implementation
- Infinite sequence of exchangeable envelopes
- Each player picks one
- It must be in his best interests to play its contents

Order k exchangeable equilibria

- What if no one could do better even looking at k envelopes?
- Tighter convex relaxation of symmetric Nash equilibria
- Converges to mixtures of symmetric Nash as $k \to \infty$
- No direct existence proof yet

Extensions of exchangeable equilibria #2

Exchangeable equilibria for asymmetric games

- Obvious generalization turns out to be trivial
  - Replace "conditionally i.i.d." with "conditionally independent"
  - $\mathrm{conv}\{xy^T \mid x, y \geq 0\} = \{X \mid X \geq 0\}$
  - Any distribution is a mixture of independent distributions
- Can do better generalizing above implementation
- Infinite exchangeable sequence of envelopes for each player
- Each player is allowed to choose one
- Best off if he chooses one of his own, plays its contents

Conclusions

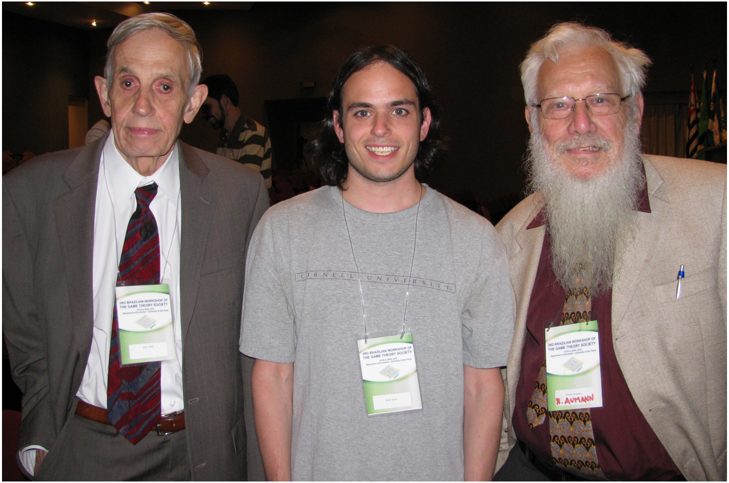
Summary

- Exchangeable equilibria: new solution concept for sym. games
- Various natural interpretations
- Between symmetric Nash and symmetric correlated equilibria
- For small games, described by a semidefinite program
- Can be approximated efficiently in general
- Generalizations give tighter relaxations, asymmetric version

Open questions

- Avoid ellipsoid method?
- Direct existence of order k exchangeable equilibria?

A final thought



$\mathrm{NE}_{\mathrm{Sym}}$	$\subseteq$	XE	$\subseteq$	$\mathrm{CE}_{\mathrm{Sym}}$
John Nash		Noah Stein		Robert Aumann
(Nobel 1994)		?		(Nobel 2005)